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ANGULAR MOTION OF SPINNING ALMOST SYMMETRIC MISSILES

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An almost symmetric missile is a missile whose zero-spin pitch and yaw frequencies are "nearly" equal. The angular motion of a spinning almost symmetric missile can be described by five rotating modal vectors. Two of these vectors vanish when the frequencies are equal. Good approximations for modal frequencies and amplitudes are derived for the special cases of near zero spin and spins near pitch or yaw resonance as well as for spins away from these special regions.

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I. INTRODUCTION

In 1970, Ward and Mansfield[1] made some hypersonic damping-in-pitch measurements of 9-degree cones by the free oscillation technique. Since an air bearing support was used, the models were free to spin as well as pitch and yaw. An apparently minor tip asymmetry was induced by a planar cut 40° off the X-Z plane. The projected frontal area of the cut was approximately 0.5% of the base area.

The free oscillations were then fitted by the Nicolaides tricyclic theory.[2] This theory assumes that the sole effect of small asymmetry is to add a missile-fixed constant-amplitude trim moment term to the moment expression for the unmodified symmetric cone. A missile that satisfies this assumption is said to have a slight configurational asymmetry. It was found, however, that the measured motion cannot be reasonably represented by the tricyclic theory. In 1972, Walchner, Sawyer, and Yelmgren[3] showed that the apparently very small asymmetry used in the AEDC tests was sufficient to make the pitch and yaw frequencies unequal and thus a more complex mathematical moment assumption is required to analyze the free oscillation data. References 4 and 5 discuss two other cases of apparently symmetric missiles whose motions seem to be poorly described by either the epicyclic motion performed by symmetric missiles or the tricyclic motion performed by slightly asymmetric missiles.

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1. *L.K. Ward and A.C. Mansfield, "Dynamic Characteristics of a 9-Degree Cone with and without Asymmetries at Mach Number 10," Arnold Engineering Development Center TR-70-1, March 1970.*

2. *J.D. Nicolaides, "On the Free Flight Motion of Missiles Having Slight Configurational Asymmetries," Ballistic Research Laboratories Report No. 858, June 1953, AD 26405. (See also IAS Preprint 395, 1952.)*

3. *O. Walchner, F.M. Sawyer and K.E. Yelmgren, "Aerodynamics of a Slender Cone with Asymmetric Nose Bluntness at Mach 14," AIAA Journal 10, August 1972, pp. 1121-1122.*

4. *P.F. Intrieri, D.B. Kirk, G.T. Chapman and J.E. Terry, "Ballistic Range Tests of Ablating and Nonablating Slender Cones," AIAA Journal 8, March 1970, pp. 558-564.*

5. *P. Jaffe, "Nonplanar Tests Using Wind Tunnel Free Flight Technique," Journal of Spacecraft and Rockets 10, July 1973, pp. 435-442.*

The effect of constant axial spin on the stability of aircraft was first considered by Phillips.[6] Recently, Hodapp[7-8] has extended this work to unsymmetric re-entry vehicles in a most elegant fashion. Both Phillips and Hodapp make use of missile-fixed coordinates. These coordinates are convenient if the missile is not spinning or if on-board instrumentation is used. For the three experiments cited earlier, external photographic instrumentation was used and the nonrolling aeroballistic axes are much more convenient. A second difficulty of the Phillips and Hodapp papers is that the general pitching and yawing motion of unsymmetric missiles is described by five modal vectors and the results are given in a form in which it is difficult to see how the five vectors reduce to the three modal vectors appropriate to symmetric motion with trim.

In this paper, we make use of the complex variables that considerably simplify the analysis of symmetric missile motion.[9] These variables allow us to move easily from missile-fixed axes to nonspinning axes. They also allow us to obtain very simple expressions for an "almost" symmetric missile. All results are obtained for constant spin.

In Sections II through VI, the analysis is limited to a linear yaw moment and a linear pitch moment with C_{m_α} not equal to $-C_{n_\beta}$. Since the pitch and the yaw frequencies are different for zero spin, there is a resonance region for spin when the spin is between these frequencies. After deriving exact relations for the two frequencies of the combined pitching and yawing motion for a spinning missile, we obtain approximations for an "almost symmetric" missile. An almost symmetric missile is one whose zero-spin pitch frequency is very near its zero-spin yaw frequency. For these approximations, three regions of spin are considered separately: spin near zero, spin near resonance, and spin far from zero spin and resonance spin. For the third region, it is shown that in the aeroballistic system, two of the frequencies are approximated by the fast and slow rates $\dot{\phi}_{1S}$, $\dot{\phi}_{2S}$ appropriate to a symmetric missile with a static moment derivative

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6. *W.H. Phillips, "Effect of Steady Rolling on Longitudinal and Directional Stability," NACA Technical Note No. 1627, June 1948.*

7. *A.E. Hodapp, Jr., "Effects of Unsymmetrical Stability Derivative Characteristics on Re-Entry Vehicle Trim Angle Behavior," Journal of Spacecraft and Rockets 11, May 1974, pp. 300-307.*

8. *A.E. Hodapp, Jr., "Effects of Unsymmetrical Stability Derivative Characteristics on Re-Entry Vehicle Transient Angular Motion," Journal of Spacecraft and Rockets 13, February 1976, pp. 82-90.*

9. *C.H. Murphy, "Free Flight Motion of Symmetric Missiles," Ballistic Research Laboratories Report No. 1216, July 1963, AD 442757.*

$$C_{M_\alpha} = (C_{m_\alpha} - C_{n_\beta})/2 \qquad (1.1)$$

Two other modes exist with frequencies $2p - \dot{\phi}_{1S}$, $2p - \dot{\phi}_{2S}$. The amplitude of these modes is shown to approach zero as $C_{m_\alpha} \to -C_{n_\beta}$.

Near resonance the fast mode and its associated frequency $2p - \dot{\phi}_1$ both approach the spin rate, p. For spin rates in the resonance zone between the pitch and yaw frequencies, both $\dot{\phi}_1$ and $2p - \dot{\phi}_1$ become equal to the spin rate and one mode is exponentially damped while the other is exponentially undamped.

In Section VII, the effect of linear aerodynamic damping is studied. Its major effect is to reduce the region of spin rates for which exponential undamping exists. Outside the resonance region, the two damping rates are very similar to those for a completely symmetric missile.

II. EQUATIONS OF MOTION

The velocity equation for nearly horizontal flight can be easily written in terms of the aerodynamic force along the trajectory--the drag force:

$$m\dot{V} = -\ (\rho V^2/2)SC_D \qquad (2.1)$$

If the independent variable is changed from time to dimensionless arc-length, s, this reduces to

$$\frac{V'}{V} = -\left(\frac{\rho S\ell}{2m}\right)C_D \equiv -C_D^* \qquad (2.2)$$

The usual coordinates to describe the angular motion of a rigid body are body-fixed coordinates. For the case of a symmetric missile, the aeroballistic axes, which pitch and yaw with the missile but have a zero spin rate, are more convenient. In this study of an "almost" symmetric missile, we will use missile-fixed axes to obtain our results and then transform the results to the aeroballistic axes.

The X-axis is fixed in the body along the principal axis of inertia nearest to the desired flight direction. Since we retain the assumption of mass rotational symmetry, the transverse moments of inertia are equal and for any roll orientation of the Y and Z axes, these axes will also be along principal axes of inertia. The actual orientation of these axes will be selected to simplify the aerodynamic moment. (The effect of small mass asymmetry is given in Appendix A.)

The equations of motion can be written in terms of the aerodynamic force and moment and the derivatives of the linear and angular momentum. (For simplicity the effect of gravity has been neglected. Its effect on the angular motion is usually quite small.)

$$m \dot{\vec{V}} = (\rho V^2/2)S \ (C_X, \ C_Y, \ C_Z) \tag{2.3}$$

$$\dot{\vec{H}} = (\rho V^2/2)S\ell \ (C_\ell, \ C_m, \ C_n) \tag{2.4}$$

where

$$\vec{V} = (u, \ v, \ w)$$

$$\vec{H} = (I_x \ p, \ I_t \ q, \ I_t \ r)$$

The vector derivatives can be computed by the well-known relation for any vector (U_x, U_y, U_z):

$$(U_x, \ U_y, \ U_z)^{\cdot} = (\dot{U}_x, \ \dot{U}_y, \ \dot{U}_z) + \vec{\omega} \times (U_x, \ U_y, \ U_z) \tag{2.5}$$

where

$$\vec{\omega} = (p, \ q, \ r) \ .$$

Equations (2.3-2.4) can be expanded by use of Equation (2.5) and the independent variable changed from t to s. For small angles ($u \sim V$), the transverse components of these vector equations become

$$\frac{v'}{V} + \frac{r\ell}{V} - \phi' \left(\frac{w}{V}\right) = C_Y^* \tag{2.6}$$

$$\frac{w'}{V} + \phi' \left(\frac{v}{V}\right) - \frac{q\ell}{V} = C_Z^* \tag{2.7}$$

$$\frac{q'\ell}{V} - \left(1 - \frac{I_x}{I_t}\right)\phi' \left(\frac{r\ell}{V}\right) = k_t^{-2} \ C_m^* \tag{2.8}$$

$$\frac{r'\ell}{V} + \left(1 - \frac{I_x}{I_t}\right)\phi' \left(\frac{q\ell}{V}\right) = k_t^{-2} \ C_n^* \tag{2.9}$$

The complex notation that is so helpful for a symmetric missile can now be introduced. The complex angle of attack and transverse angular velocity have the usual definitions:

$$\xi = \frac{v + i\,w}{V} \doteq \beta + i\,\alpha \tag{2.10}$$

$$\mu = (q + i\,r)\,\ell/V \tag{2.11}$$

By differentiating Equations (2.10) and (2.11) we obtain (with the aid of Equations (2.2) and (2.6-2.9)):

$$\xi' + i\,\phi'\xi - i\,\mu = (C_Y^* + i\,C_Z^*) + C_D^*\xi \tag{2.12}$$

$$\mu' + i\,(\phi' - P)\mu = k_t^{-2}\,(C_m^* + i\,C_n^*) + C_D^*\,\mu \tag{2.13}$$

where

$$P = (I_x/I_t)\phi'$$

$$k_t^2 = I_t/m\ell^2$$

III. LINEAR STATIC MOMENT

The most general moment components that are linear in α and β are

$$C_m = C_{m_0} + C_{m_\alpha}\,\alpha + C_{m_\beta}\,\beta \tag{3.1}$$

$$C_n = C_{n_0} + C_{n_\alpha}\,\alpha + C_{n_\beta}\,\beta \tag{3.2}$$

In complex form, this becomes

$$C_m + i\,C_n = C_{m_0} + i\,C_{n_0} + \left[C_{MS_\alpha} - i\,C_{M_\alpha}\right]\xi$$

$$+ \left[\hat{C}_{MS_\alpha} + i\,\hat{C}_{M_\alpha}\right]\bar{\xi} \qquad (3.3)$$

where

$$C_{M_\alpha} = \left(C_{m_\alpha} - C_{n_\beta}\right)/2$$

$$\hat{C}_{M_\alpha} = \left(C_{m_\alpha} + C_{n_\beta}\right)/2$$

$$C_{MS_\alpha} = \left(C_{m_\beta} + C_{n_\alpha}\right)/2$$

$$\hat{C}_{MS_\alpha} = \left(C_{m_\beta} - C_{n_\alpha}\right)/2$$

The effect of rotating the Y and Z axes through the angle θ_0 can be easily calculated by multiplying Equation (3.3) by exp $(i\theta_0)$.

$$\left[C_m + i\,C_n\right]e^{i\theta_0} = \left[C_{m_0} + i\,C_{n_0}\right]e^{i\theta_0}$$

$$+ \left[C_{MS_\alpha} - i\,C_{M_\alpha}\right]\left[\xi\,e^{i\theta_0}\right]$$

$$+ \left[\left(\hat{C}_{MS_\alpha} + i\,\hat{C}_{M_\alpha}\right)e^{2i\theta_0}\right]\left[\overline{\xi\,e^{i\theta_0}}\right]$$

$$(3.4)$$

According to Equation (3.4), the coefficient of the complex angle is unaffected by a rotation of coordinates but the coefficient of the conjugate of the complex angle is multiplied by exp $(2i\theta_0)$. This property allows us to choose a convenient orientation of the Y and Z axes. This orientation will be selected so that the coefficient of the conjugate complex angle is zero or a negative imaginary number ($\hat{C}_{MS_\alpha} = 0$, $\hat{C}_{M_\alpha} \leq 0$). In terms of the coefficients defined in Equations (3.1-3.2),

$$C_{m_\beta} = C_{n_\alpha}$$

$$-C_{m_\alpha} \geqq C_{n_\beta}$$

For a symmetric missile, $\hat{C}_{M_\alpha}$ is zero and C_{MS_α} is an odd function of spin. (The side moment for this case is usually called a Magnus moment.) The addition of a complex constant $(C_{m_0} + i\, C_{n_0})$ to the moment of a symmetric missile introduces a trim angle and the possibility of resonance.[2, 10-11]

The primary effect of the C_{MS_α} side moment coefficient on the angular motion of a symmetric missile is on the damping rates and not on the frequencies. The damping rates are assumed to be small compared with the frequencies and will be calculated later as a perturbation of the zero-damped motion. Thus, the side moment will be grouped with the aerodynamic forces and the damping moments and neglected until the perturbations are introduced. Hence the static moment to be used in the first stage of our analysis is defined by

$$\begin{aligned} C_m + i\, C_n &= C_{m_0} + i\, C_{n_0} - i\left[C_{M_\alpha}\, \xi - \hat{C}_{M_\alpha}\, \bar{\xi}\right] \\ &= C_{m_0} + C_{m_\alpha}\, \alpha + i\left[C_{n_0} + C_{n_\beta}\, \beta\right] \end{aligned} \qquad (3.5)$$

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10. *R. L. Nelson, "The Motions of Rolling Symmetrical Missiles Referred to Body-Axis System," NACA Technical Note No. 1627, November 1956.*

11. *D.A. Price, Jr., and L. E. Ericsson, "A New Treatment of Roll-Pitch Coupling for Ballistic Re-Entry Vehicles," AIAA Journal 8, September 1970, pp. 1608-1615.*

IV. FREQUENCIES AND MODAL AMPLITUDES

The static moment of Equation (3.5) can now be inserted in Equation (2.13), the aerodynamic force neglected ($C_D = C_Y = C_Z = 0$) and Equation (2.12) used to eliminate μ . We obtain

$$\xi'' + i\, f\, \phi'\, \xi' - M_r\, \xi + \hat{M}\, \bar{\xi} = i\, A \tag{4.1}$$

where

$$f = 2 - \frac{I_x}{I_t}$$

$$M_r = M + (f - 1)(\phi')^2$$

$$M = k_t^{-2}\, C^*_{M_\alpha}$$

$$\hat{M} = k_t^{-2}\, \hat{C}^*_{M_\alpha} \leqq 0$$

$$A = k_t^{-2} \left(C^*_{m_0} + i\, C^*_{n_0} \right)$$

Since Equation (4.1) is a fourth-order differential system in the real variables α and β, periodic solutions correspond to two pairs of conjugate roots of a fourth-degree polynomial.

For a symmetric missile with trim, $\hat{M}$ is zero and the solution to Equation (4.1) can be written in terms of two rotating complex vectors plus a constant trim vector.

$$\xi = k_1\, e^{i\omega_1 s} + k_2\, e^{i\omega_2 s} + k_3 \tag{4.2}$$

where

$$\omega_j = \frac{1}{2}\left[- f\,\phi' \pm [(f\,\phi')^2 - 4\,M_r]^{\frac{1}{2}}\right]$$

$$= -\,\phi' + \frac{1}{2}\left[P \pm [P^2 - 4M]^{\frac{1}{2}}\right]$$

$$k_3 = -\,i\,A/M_r$$

In the nonrolling aeroballistic coordinates, this becomes

$$\tilde{\xi} = \xi\, e^{i\phi' s} = k_1\, e^{i\phi_1' s} + k_2\, e^{i\phi_2' s} + k_3\, e^{i\phi' s} \tag{4.3}$$

where

$$\phi_j' = \omega_j + \phi' = \left[P \pm [P^2 - 4M]^{\frac{1}{2}}\right]\Big/2$$

Note that the trim vector k_3 becomes infinite for $M_r = 0$. This is the usual spin-yaw resonance condition and the spin that makes M_r vanish is resonance spin.

Usually ϕ_1' is defined to be the larger frequency and, if we limit the spin to positive values, ω_1 would then correspond to the positive square root and ω_2 to the negative square root. This means that for negative M, ω_2 is always negative and ω_1 is positive for M_r negative and negative for M_r positive. Thus, ω_1 changes sign at resonant spin and is always less in magnitude than ω_2 .

Since Equation (4.1) contains $\bar{\xi}$, the solution should consist of complex vectors and their conjugates.

$$\xi = k_1\, e^{i\omega_1 s} + k_2\, e^{i\omega_2 s} + k_3$$

$$+ k_4\, e^{-i\omega_1 s} + k_5\, e^{-i\omega_2 s} \tag{4.4}$$

By direct substitution into Equation (4.1), we see that the trim vector takes the very simple form

$$k_3 = \frac{- i[M_r A - \hat{M} \bar{A}]}{M_r^2 - \hat{M}^2} \tag{4.5}$$

Resonance now occurs for the two values of spin that make the denominator in Equation (4.5) vanish. In terms of pitch and yaw zero-spin-rate frequencies ω_α and ω_β , these spin rates are

$$\phi'_\ell = \omega_\ell [f - 1]^{-\frac{1}{2}} \qquad \ell = \alpha, \beta \tag{4.6}$$

where

$$\omega_\alpha = [- k_t^{-2} \ C^*_{m_\alpha}]^{\frac{1}{2}}$$

$$\omega_\beta = [k_t^{-2} \ C^*_{n_\beta}]^{\frac{1}{2}}$$

Note that the pitch and yaw axes were selected so that

$$\phi'_\alpha \geq \phi'_\beta \tag{4.7}$$

Direct substitution of Equation (4.4) into Equation (4.1) yields two pairs of equations, for k_1 and k_4 and for k_2 and k_5 . The first pair can be written in the form

$$k_1 \ [C_1 + 2 \ f \ \phi' \ \omega_1] - \bar{k}_4 \ \hat{M} = 0 \tag{4.8}$$

$$- \bar{k}_1 \ \hat{M} + k_4 \ \ C_1 \ \ = 0 \tag{4.9}$$

where

$$C_1 \ = \omega_1^2 - f \ \phi' \ \omega_1 + M_r \tag{4.10}$$

The solution to Equations (4.8-4.9) is

$$k_4 = \hat{M} \ \bar{k}_1 \ C_1^{-1} \tag{4.11}$$

$$C_1^2 + 2\, f\, \phi'\, \omega_1\, C_1 - \hat{M}^2 = 0 \tag{4.12}$$

Similar equations follow for k_2, k_5, C_2 and ω_2. Equation (4.12) and the corresponding equation for C_2 can be solved:

$$C_j = -\, f\, \phi'\, \omega_j \pm [(f\, \phi' \omega_j)^2 + \hat{M}^2]^{\frac{1}{2}} \tag{4.13}$$

Since k_1 and k_2 are the only modes present for a symmetric missile, k_4 and k_5 should vanish for $\hat{M} = 0$ and therefore C_j should not be zero for this value of $\hat{M}$. This means that the plus sign in Equation (4.13) is appropriate for C_1 when $\phi' > \phi'_\alpha$ and for C_2 for all spins. The negative sign should be used for C_1 when $\phi' < \phi'_\beta$. This can all be summarized as follows:

$$\left.\begin{aligned} C_1 &= -\, f\, \phi' \omega_1 + \delta\, [(f\, \phi'\, \omega_1)^2 + \hat{M}^2]^{\frac{1}{2}} \\ C_2 &= -\, f\, \phi'\, \omega_2 + [(f\, \phi'\, \omega_2)^2 + \hat{M}^2]^{\frac{1}{2}} \end{aligned}\right\} \tag{4.14}$$

where

$$\delta = \begin{cases} 1 \text{ if } \phi' > \phi'_\alpha \\ -1 \text{ if } 0 \leq \phi' < \phi'_\beta \end{cases}$$

The frequency equations follow from Equations (4.10, 4.12) and the corresponding equations for the second frequency.

$$\omega_j^4 - 2\, a\, \omega_j^2 + b = 0 \tag{4.15}$$

$$j = 1, 2$$

where

$$a = (1/2)(f\ \phi')^2 - M_r$$

$$b = M_r^2 - \hat{M}^2$$

$$= (f - 1)^2[(\phi')^2 - (\phi'_\alpha)^2][(\phi')^2 - (\phi'_\beta)^2]$$

The solution to Equation (4.15) can be written in the form:

$$\omega_j^2 = a \pm \sqrt{a^2 - b}$$

$$= \left[\left(\frac{a + \sqrt{b}}{2}\right)^{1/2} \pm \left(\frac{a - \sqrt{b}}{2}\right)^{1/2}\right]^2 \qquad (4.16)$$

Since a is always posit:,e, Equation (4.16) gives positive values of ω_1^2 and ω_2^2 for b and $(a^2 - b)$ positive. For negative b, only ω_2^2 is positive and a real ω_1 does not exist. Under these conditions, the precise relations for real ω_j are:

$$\omega_1 = \delta\left[-\left(\frac{a + \sqrt{b}}{2}\right)^{1/2} + \left(\frac{a - \sqrt{b}}{2}\right)^{1/2}\right] \qquad (4.17)$$

$$\omega_2 = -\left(\frac{a + \sqrt{b}}{2}\right)^{1/2} - \left(\frac{a - \sqrt{b}}{2}\right)^{1/2} \qquad (4.18)$$

If Equation (4.11) and its k_5 counterpart are substituted in Equation (4.4) and if the k_j's are expressed in the polar form $K_j \exp(i\ \phi_{j0})$, the solution motion takes a very simple form in aeroballistic coordinates:

$$\tilde{\xi} = K_1\ [e^{i\phi_1} + \hat{M}\ C_1^{-1}\ e^{i(2\phi - \phi_1)}]$$
$$+ K_2\ [e^{i\phi_2} + \hat{M}\ C_2^{-1}\ e^{i(2\phi - \phi_2)}] \qquad (4.19)$$
$$+ k_3\ e^{i\phi}$$

for $0 \leq \phi' < \phi'_\beta$ or $\phi' > \phi'_\alpha$, where

$$\phi_j = \phi_{j0} + \phi'_j s \qquad j = 1, 2$$

$$\phi'_j = \omega_j + \phi'$$

$$\phi = \phi' s$$

V. RESONANCE REGION

When the spin rate ϕ' is between ϕ'_β and ϕ'_α , b is negative and Equation (4.17) is invalid. (It should be noted that Equation (4.18) does give a real number for ω_2 although the calculation involves complex numbers.) For this resonance region, the terms with ω_1 in Equation (4.4) must be modified:

$$\xi = k_{1R} e^{\lambda s} + k_{4R} e^{-\lambda s} + k_2 \, e^{i\omega_2 s} + k_5 \, e^{-i\omega_2 s} + k_3 \tag{5.1}$$

where

$$\lambda > 0$$

$$k_{jR} = K_{jR} \exp\,(i\,\Phi_{jR}), \quad j = 1, 4$$

Substitution of Equation (5.1) in Equation (4.1) yields two pairs of real equations for the coefficients of exp (λs) and exp ($-\lambda$s).

$$(\lambda^2 - M_r + \hat{M}) \cos \Phi_{1R} - f\,\phi'\,\lambda \sin \Phi_{1R} = 0 \tag{5.2}$$

$$f\,\phi'\,\lambda \cos \Phi_{1R} + (\lambda^2 - M_r - \hat{M}) \sin \Phi_{1R} = 0 \tag{5.3}$$

$$(\lambda^2 - M_r + \hat{M}) \cos \Phi_{4R} + f\,\phi'\,\lambda \sin \Phi_{4R} = 0 \tag{5.4}$$

$$-\,f\,\phi'\,\lambda \cos \Phi_{4R} + (\lambda^2 - M_r - \hat{M}) \sin \Phi_{4R} = 0 \tag{5.5}$$

According to these equations,

$$\tan \Phi_{1R} = - \tan \Phi_{4R}$$

$$= \frac{-f\ \phi'\ \lambda}{\lambda^2 - M_r - \hat{M}} \tag{5.6}$$

$$\lambda^4 + 2a\ \lambda^2 + b = 0 \tag{5.7}$$

The real positive solution to Equation (5.7) is

$$\lambda = [\, - a + (a^2 - b)^{\frac{1}{2}}]^{\frac{1}{2}} \tag{5.8}$$

For an almost symmetric missile ($\hat{M} << M$), $b << a^2$ and

$$a \doteq (f\ \phi')^2/2 \tag{5.9}$$

$$\lambda \doteq [\hat{M}^2 - M_r^2]^{\frac{1}{2}}/(f\ \phi') = \sqrt{-b}/(f\ \phi') \tag{5.10}$$

$$\tan \Phi_{1R} \doteq \left[\frac{(\phi')^2 - (\phi'_\beta)^2}{(\phi'_\alpha)^2 - (\phi')^2}\right]^{\frac{1}{2}} \tag{5.11}$$

According to these approximations, λ vanishes for spin on the boundaries of the resonance region and is a maximum near the center ($M_r = 0$). The planes of the exponentially growing and decaying angles are near the pitch plane for spin near ϕ'_α and are near the yaw plane for spin near ϕ'_β. For maximum growth/decay rates, the planes of the growing angle and the decaying angle are perpendicular to each other and both planes bisect the angle between the pitch and yaw planes.

VI. C_j AND ω_j FOR ALMOST SYMMETRIC MISSILES

We will now obtain special relations for an almost symmetric missile for which $\hat{M}$ is much smaller than M. For this case, the radical in Equations (4.14) for C_j can be expanded for three regions of spin--near zero, near resonance and otherwise. These expansions yield the results in Table I.

Far from resonance, $M_r^2 >> \hat{M}^2$ and a good estimate of the frequencies can be obtained from the approximation

$$\sqrt{b} \doteq \delta\ (M_r - \hat{M}^2/2M_r) \qquad (6.1)$$

Equations (4.17—4.18) can now be expanded in terms of $\hat{M}$ to yield:

$$\phi'_j = \phi'_{jS} + g_j\ \hat{M}^2 \qquad (6.2)$$

$$\phi'_{jS} = \frac{1}{2}\ [P \pm \sqrt{P^2 - 4M}\ ] \qquad (6.3)$$

where the g_j's are given in Table II. According to these equations, very good approximations to the frequencies of an almost symmetric missile are the symmetric frequencies of a missile with the average moment coefficient C_{M_α} of the almost symmetric missile. This is correct to within a term in $\hat{M}^2$. Note that the coefficients g_j are unbounded for zero spin but only g_1 has difficulty near resonance. (g_2 has the limit $-\ [(f\ \phi')^{-3}]/2$ as $M_r \rightarrow 0$.)

Near resonance, Equation (4.17) can be expanded for small b to give the relation in Table II. A little additional algebra yields the expansion for frequencies near zero spin given in Table II.

According to this table, ϕ'_1 at zero spin is the zero-spin yaw frequency and ϕ'_1 increases as the spin increases. At frequencies well away from zero or ϕ'_β , ϕ'_1 is well approximated by ϕ'_{1S} . In the resonance region, it is the spin frequency and at spins well above the resonance region, it is once again well approximated by the symmetric missile fast frequency.

Table I. Approximations for C_j

ϕ'	C_1	C_2
$0 \leq \phi' < \varepsilon$	$\hat{M} - f\,\phi'\,\omega_1$	$-\hat{M} - f\,\phi'\,\omega_2$
$\phi'_\beta - \varepsilon < \phi' \leq \phi'_\beta$	$\hat{M} - f\,\phi'\,\omega_1$	B_2
$\phi'_\alpha \leq \phi' < \phi'_\alpha + \varepsilon$	$-\hat{M} - f\,\phi'\,\omega_1$	B_2
otherwise	B_1	B_2

where $0 < \varepsilon << \phi'_\alpha - \phi'_\beta$

$$B_j = -\,2\,f\,\phi'\,\omega_j - \hat{M}^2\,[2\,f\,\phi'\,\omega_j]^{-1}\;, \quad j = 1, 2$$

Table II. Frequencies for an Almost Symmetric Missile

ϕ'	ϕ'_1	ϕ'_2
0	ω_β	$-\omega_\alpha$
$0 < \phi' < \epsilon$	$\omega_\beta + \phi' + h\,(\phi')^2$	$-\omega_\alpha + \phi' + h\,(\phi')^2$
$0 << \phi' << \phi'_\beta$	$\phi'_{1S} + g_1\,\hat{M}^2$	$\phi'_{2S} + g_2\,\hat{M}^2$
$\phi'_\beta - \epsilon < \phi' \le \phi'_\beta$	$\phi' + [M_r^2 - \hat{M}^2]^{\frac{1}{2}}\,(f\,\phi')^{-1}$	$\phi'_{2S} + g_2\,\hat{M}^2$
$\phi'_\beta \le \phi' \le \phi'_\alpha$	ϕ'	$\phi'_{2S} + g_2\,\hat{M}^2$
$\phi'_\alpha \le \phi' < \phi'_\alpha + \epsilon$	$\phi' - [M_r^2 - \hat{M}^2]^{\frac{1}{2}}\,(f\,\phi')^{-1}$	$\phi'_{2S} + g_2\,\hat{M}^2$
$\phi'_\alpha << \phi'$	$\phi'_{1S} + g_1\,\hat{M}^2$	$\phi'_{2S} + g_2\,\hat{M}^2$

where

$$0 < \epsilon << \phi'_\alpha - \phi'_\beta$$

$$h = (-M)^{\frac{1}{2}}\,f/4\hat{M}$$

$$g_1 = -\,\omega_{2S}\,[2\,f\,\phi'\,M_r\,(\omega_{1S} - \omega_{2S})]^{-1}$$

$$g_2 = \omega_{1S}\,[2\,f\,\phi'\,M_r\,(\omega_{1S} - \omega_{2S})]^{-1}$$

ϕ_2' has a much simpler behavior. At zero spin, it is the negative of the zero-spin pitch frequency and increases as spin increases. Well away from zero spin, it is well approximated by ϕ_{2S}' .

In Table III, the various approximations for C_j are used to construct expressions for the various motions for an almost symmetric missile with no trim moment ($k_3 = 0$). This catalog of motions starts at zero spin with the expected Lissajous figure formed by combining the pitching and yawing motion with different frequencies (Case A). For slow spin, we have the more complicated form of Case B.

Case C is the general case for spin well away from zero or resonance. It is a four-mode motion with two modes quite similar to those associated with the symmetric missile epicyclic motion with frequencies ϕ_1' and ϕ_2' . The two additional modes have much smaller amplitudes that go to zero as the missile becomes symmetric ($\hat{M} \to 0$). Their frequencies have the rather odd form $2\phi' - \phi_j'$.

As the spin approaches the resonance region, ϕ_1' and $2\phi' - \phi_1'$ become equal and the motion associated with K_1 changes character. The amplitude of the $2\phi' - \phi_1'$ mode approaches the amplitude of the ϕ_1' mode (Case D). At the singular spin value of ϕ_β' , a very special secular solution is found (Case E). In the resonance region (Case F), the two modes associated with ϕ_1' both rotate at the spin rate but one mode grows exponentially and the other decays exponentially. The exponent, λ, depends on the size of the resonance region and how far the spin rate is from the boundaries. Case G is the secular solution for the upper boundary of the resonance region and Case H is the motion for slightly higher spins.

In order to show this spectrum of motions in a more specific way, the frequencies, modal amplitudes and resonance damping rate have been computed for an almost symmetric missile with $\hat{C}_{M_\alpha} = 0.095\ C_{M_\alpha}$ and $f = 1.9$. For these values,

$$\phi_\alpha'/\phi_\beta' = 1.10$$

$$\omega_\alpha/\phi_\beta' = 1.04$$

$$\omega_\beta/\phi_\beta' = 0.95 \quad .$$

Table III. Aeroballistic Motion

$\boxed{\text{A.} \quad \phi' = 0}$

$$\tilde{\xi} = \xi = 2 K_1 \cos \phi_1 + 2 i K_2 \sin \phi_2$$

$\boxed{\text{B.} \quad 0 < \phi' < \varepsilon}$

$$\tilde{\xi} = K_1 \left[e^{i\phi_1} + [1 - f \phi' \omega_1 \hat{M}^{-1}]^{-1} e^{i(2\phi - \phi_1)} \right] + K_2 \left[e^{i\phi_2} - [1 + f \phi' \omega_2 \hat{M}^{-1}]^{-1} e^{i(2\phi - \phi_2)} \right]$$

$\boxed{\text{C.} \quad 0 << \phi' << \phi'_\beta, \ \phi' >> \phi'_\alpha}$

$$\tilde{\xi} = K_1 \left[e^{i\phi_1} - \hat{M} (2 f \phi' \omega_1)^{-1} e^{i(2\phi - \phi_1)} \right] + K_2 \left[e^{i\phi_2} - \hat{M} (2 f \phi' \omega_2)^{-1} e^{i(2\phi - \phi_2)} \right]$$

$\boxed{\text{D.} \quad \phi'_\beta - \varepsilon < \phi' < \phi'_\beta}$

$$\tilde{\xi} = K_1 \left[e^{i\phi_1} + [1 - f \phi' \omega_1 \hat{M}^{-1}]^{-1} e^{i(2\phi - \phi_1)} \right] + K_2 \left[e^{i\phi_2} - \hat{M} (2 f \phi' \omega_2)^{-1} e^{i(2\phi - \phi_2)} \right]$$

$\boxed{\text{E.} \quad \phi' = \phi'_\beta}$

$$\tilde{\xi} = K_{1\beta} e^{i\phi} \left[\cos \phi_\beta + [i + 2 \hat{M} (f \phi')^{-1} s] \sin \phi_\beta \right] + K_2 \left[e^{i\phi_2} - \hat{M} (2 f \phi' \omega_2)^{-1} e^{i(2\phi - \phi_2)} \right]$$

$\boxed{\text{F.} \quad \phi'_\beta < \phi' < \phi'_\alpha \quad \text{(Resonance Region)}}$

$$\tilde{\xi} = e^{i\phi} \left[K_{1R} e^{\lambda s + i \phi_{1R}} + K_{4R} e^{-\lambda s - i\phi_{1R}} \right] + K_2 \left[e^{i\phi_2} - \hat{M} (2 f \phi' \omega_2)^{-1} e^{i(2\phi - \phi_2)} \right]$$

$\boxed{\text{G.} \quad \phi' = \phi'_\alpha}$

$$\tilde{\xi} = K_{1\alpha} e^{i\phi} \left[[1 + 2 i \hat{M} (f \phi')^{-1} s] \cos \phi_\alpha + i \sin \phi_\alpha \right] + K_2 \left[e^{i\phi_2} - \hat{M} (2 f \phi' \omega_2)^{-1} e^{i(2\phi - \phi_2)} \right]$$

$\boxed{\text{H.} \quad \phi'_\alpha < \phi' < \phi'_\alpha + \varepsilon}$

$$\tilde{\xi} = K_1 \left[e^{i\phi_1} - [1 + f \phi' \omega_1 \hat{M}^{-1}]^{-1} e^{i(2\phi - \phi_1)} \right] + K_2 \left[e^{i\phi_2} - \hat{M} (2 f \phi' \omega_2)^{-1} e^{i(2\phi - \phi_2)} \right]$$

$\boxed{\text{I.} \quad \phi' >> \phi'_\alpha}$

Same as Case C

where

$$\tan \phi_{1R} = \left\{ [(\phi')^2 - (\phi'_\beta)^2] \big/ [(\phi'_\alpha)^2 - (\phi')^2] \right\}^{1/2}$$

and where K_{1R}, K_{4R}, ϕ_α, ϕ_β, $K_{1\alpha}$, $K_{1\beta}$ are parameters determined by initial conditions.

Figures 1 and 2 give the actual frequencies as functions of ϕ'/ϕ'_β and compare them with the corresponding frequencies for a symmetric missile with moment coefficient C_{M_α}. ϕ'_1 is essentially ϕ'_{1S} except near zero spin and resonance and ϕ'_2 is essentially ϕ'_{2S} except near zero spin. At zero spin, ϕ'_1 and ϕ'_2 reduce to ω_β and $-\omega_\alpha$, respectively.

Figures 3 and 4 show the variation of the "asymmetric" modal amplitudes K_4 and K_5 in comparison with the corresponding "symmetric" modal amplitudes K_1 and K_2. K_4 is quite small except near zero spin and near resonance and K_5 is quite small except near zero spin. It is important to note that K_4 is always a larger fraction of K_1 than K_5 is of K_2. This is due to the presence of ω_j in C_j and the fact that ω_1 is always smaller in amplitude than ω_2. Thus we would expect to observe the presence of the $2\phi' - \phi'_1$ mode before the $2\phi' - \phi'_2$ mode has a noticeable effect.

Finally, the resonance damping rate is plotted in Figure 5. The peak value of .047 corresponds to about a 30% increase in amplitude during one revolution.

VII. EFFECT OF DAMPING

In order to complete our study of an almost symmetric missile, we will now assume asymmetric normal force and damping moments in addition to the side moment we omitted in section III. These terms are generalizations of those of Reference 9.

$$C_Y + i\, C_Z = -\, C_{N_\alpha}\, \xi - \hat{C}_{N_\alpha}\, \bar{\xi} \tag{7.1}$$

$$\begin{aligned} C_m + i\, C_n = {} & C_{m_0} + i\, C_{n_0} + [C_{MS_\alpha} - i\, C_{M_\alpha}]\xi \\ & + i\, \hat{C}_{M_\alpha}\, \bar{\xi} + C_{M_q}\, \mu + \hat{C}_{M_q}\, \bar{\mu} \\ & - i\, C_{M_{\dot{\alpha}}}\, [\xi' + i\, \phi' \xi] \\ & + i\, \hat{C}_{M_{\dot{\alpha}}}\, [\bar{\xi}' - i\, \phi'\, \bar{\xi}] \end{aligned} \tag{7.2}$$

The new coefficients are defined in Table IV. This assumed force and moment can now be inserted in Equations (2.12-2.13) and μ eliminated between the resulting equations. (Due to size considerations, products of C* terms are ignored.) We obtain

$$\begin{aligned} & \xi'' + [H + i\, f\, \phi']\xi' - [M_r + i(M_S - \phi' H)]\xi \\ & = i\, A - \hat{H}\, \bar{\xi}' - [\hat{M} + i\, (\hat{M}_S - \phi'\, \hat{H})]\bar{\xi} \end{aligned} \tag{7.3}$$

where

$$H = C^*_{N_\alpha} - 2\, C^*_D - k_t^{-2}\, (C^*_{M_q} + C^*_{M_{\dot{\alpha}}})$$

$$M_S = P\, (C^*_{N_\alpha} - C^*_D) + k_t^{-2}\, C^*_{MS_\alpha}$$

$$\hat{H} = \hat{C}^*_{N_\alpha} + k_t^{-2}\, (\hat{C}^*_{M_q} + \hat{C}^*_{M_{\dot{\alpha}}})$$

$$\hat{M}_S = f\, \phi'\, \hat{C}^*_{N_\alpha}$$

Table IV. Aerodynamic Coefficients

$$C_{N_\alpha} = -(C_{Y_\beta} + C_{Z_\alpha})/2 \qquad \hat{C}_{N_\alpha} = -(C_{Y_\beta} - C_{Z_\alpha})/2$$

$$C_{M_\alpha} = (C_{m_\alpha} - C_{n_\beta})/2 \qquad \hat{C}_{M_\alpha} = (C_{m_\alpha} + C_{n_\beta})/2$$

$$C_{M_{\dot\alpha}} = (C_{m_{\dot\alpha}} - C_{n_{\dot\beta}})/2 \qquad \hat{C}_{M_{\dot\alpha}} = (C_{m_{\dot\alpha}} + C_{n_{\dot\beta}})/2$$

$$C_{M_q} = (C_{m_q} + C_{n_r})/2 \qquad \hat{C}_{M_q} = (C_{m_q} - C_{n_r})/2$$

$$C_{MS_\alpha} = (C_{m_\beta} + C_{n_\alpha})/2 \qquad \hat{C}_{MS_\alpha} = (C_{m_\beta} - C_{n_\alpha})/2$$

where

$$C_Y = C_{Y_\beta}\,\beta$$

$$C_Z = C_{Z_\alpha}\,\alpha$$

$$C_m = C_{m_0} + C_{m_\beta}\beta + C_{m_\alpha}\alpha + C_{m_{\dot\alpha}}(\alpha' + \phi'\beta) + C_{m_q}(q\ell/V)$$

$$C_n = C_{n_0} + C_{n_\beta}\beta + C_{n_\alpha}\alpha + C_{n_{\dot\beta}}(\beta' - \phi'\alpha) + C_{n_r}(r\ell/V)$$

The solution is assumed to have the form:

$$\xi = k_1 e^{(\lambda_1 + i\,\omega_1)s} + k_2\, e^{(\lambda_2 + i\,\omega_2)s}$$

$$+ k_3 + k_4\, e^{(\lambda_1 - i\,\omega_1)s} + k_5\, e^{(\lambda_2 - i\,\omega_2)s} \tag{7.4}$$

Substituting in Equation (7.3), we have

$$k_3 = \frac{-i\left\{[M_r - i\,(M_S - \phi'\,H)]\,A - [\hat{M} + i\,(\hat{M} - \phi'\hat{H})]\bar{A}\right\}}{M_r^2 - \hat{M}^2 + (M_S - \phi'\,H)^2 - (\hat{M}_S - \phi'\,\hat{H})^2} \tag{7.5}$$

$$k_1\left[C_1 + 2\,f\,\phi'\,\omega_1 - i\,(2\,\phi_1' - P)(\lambda_1 - \lambda_{1S})\right]$$

$$- \bar{k}_4\left[\hat{M} + i\,[(\omega_1 - \phi')\hat{H} + \hat{M}_S]\right] = 0 \tag{7.6}$$

$$- k_1\left[\hat{M} + i\,[(\omega_1 + \phi')\hat{H} - \hat{M}_S]\right]$$

$$+ \bar{k}_4\left[C_1 + i[(2\phi_1' - P)(\lambda_1 - \lambda_{1S})\right.$$

$$\left.- 2\,\omega_1\,(H + 2\,\lambda_1\,)]\right] = 0 \tag{7.7}$$

where

$$\lambda_{jS} = \frac{-[H\phi_j' - M_S]}{2\phi_j' - P}$$

and similar equations hold for k_2 and k_5. Outside the resonance region, the real part of the determinate of the coefficients in Equations (7.6-7.7) set equal to zero gives a slightly modified relation for the frequency ϕ'_1. The imaginary part of this determinate set equal to zero gives a good approximation for the damping. With the approximation

$$C_1 \doteq -2 f \phi' \omega_1 \tag{7.8}$$

the damping exponent has the following simple approximation for spins far from zero and the resonance region:

$$\lambda_1 = \lambda_{1S} + \frac{\hat{M} \hat{H}}{f \phi' (2 \phi'_1 - P)} \tag{7.9}$$

Similarly

$$\lambda_2 = \lambda_{2S} + \frac{\hat{M} \hat{H}}{f \phi' (2 \phi'_2 - P)} \tag{7.10}$$

Thus the damping exponent differs only a little from the damping for a symmetric missile with the average moment coefficients.

The bounds of the resonance region are set by the zeros of the denominator of Equation (7.5). If we neglect the side moment coefficient, this condition has the simple form.

$$M_r^2 - \hat{M}^2 + (\phi')^2 F = 0 \tag{7.11}$$

where

$$F = [H - (2 - f)(C^*_{N_\alpha} - C^*_D)]^2$$

$$- [\hat{H} - f \hat{C}^*_{N_\alpha}]^2$$

If we denote the damping-modified resonance spins as $\phi'_{\beta R}$ and $\phi'_{\alpha R}$ and assume F to be small, the modified resonance spins can be computed as perturbations of the zero-damping resonance spins ϕ'_β and ϕ'_α.

$$\phi'_{\beta R} = \left[1 - \frac{F}{4\ (f - 1)\ \hat{M}}\right] \phi'_{\beta} \qquad (7.12)$$

$$\phi'_{\alpha R} = \left[1 + \frac{F}{4\ (f - 1)\ \hat{M}}\right] \phi'_{\alpha} \qquad (7.13)$$

Since $\hat{M}$ was selected to be negative, a positive F reduces the size of the resonance region and negative F increases it.

VIII. SUMMARY

1. The general motion of an almost symmetric missile is well approximated by a symmetric missile with average coefficients.

2. Far from zero spin or resonance spin rates, the first observable modification of the usual tricyclic motion for an almost symmetric missile is the appearance of a $2\phi' - \phi'_1$ frequency followed by the appearance of a $2\phi' - \phi'_2$ frequency as the asymmetry becomes greater.

3. Near zero spin, both of these additional frequencies have substantial amplitudes and near resonance, the $2\phi' - \phi'_1$ frequency has a substantial amplitude.

4. For spin in the resonance region, large trims and exponential undamping are possible.

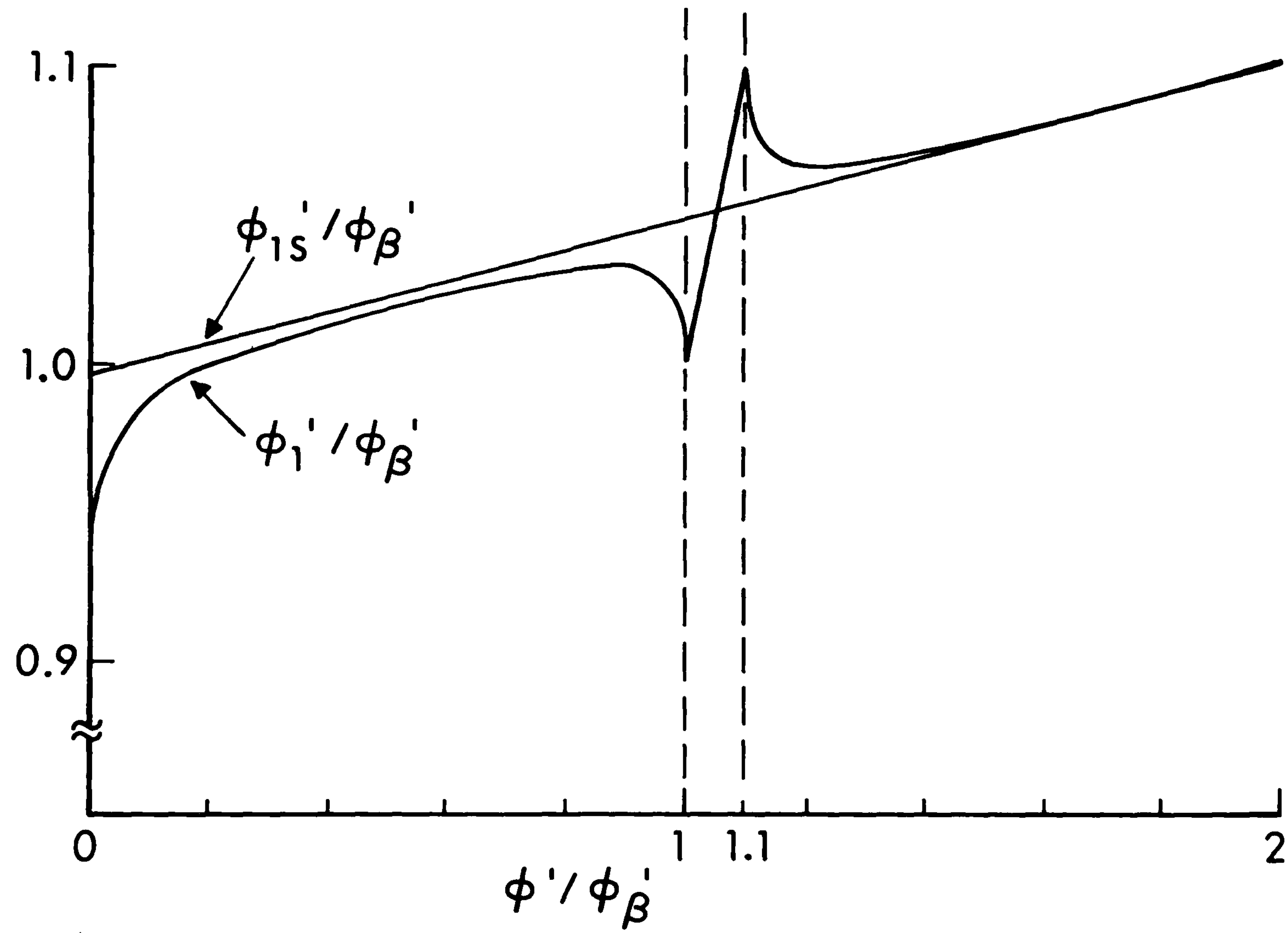

Figure 1. Frequencies ϕ_1'/ϕ_β' and ϕ_{1S}'/ϕ_β' versus Spin Rate ϕ'/ϕ_β'

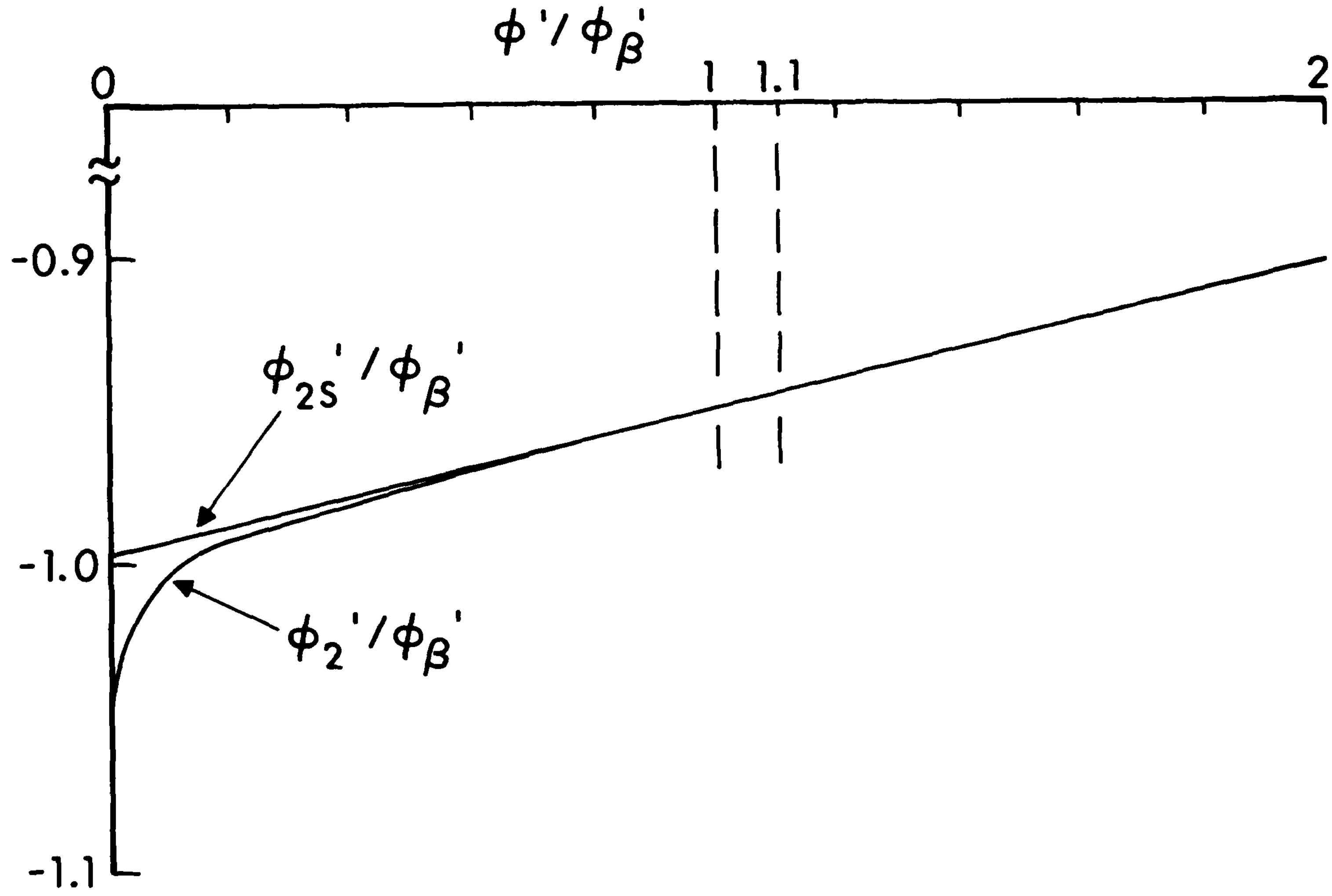

Figure 2. Frequencies ϕ_2'/ϕ_β' and ϕ_{2S}'/ϕ_β' versus Spin Rate ϕ'/ϕ_β'

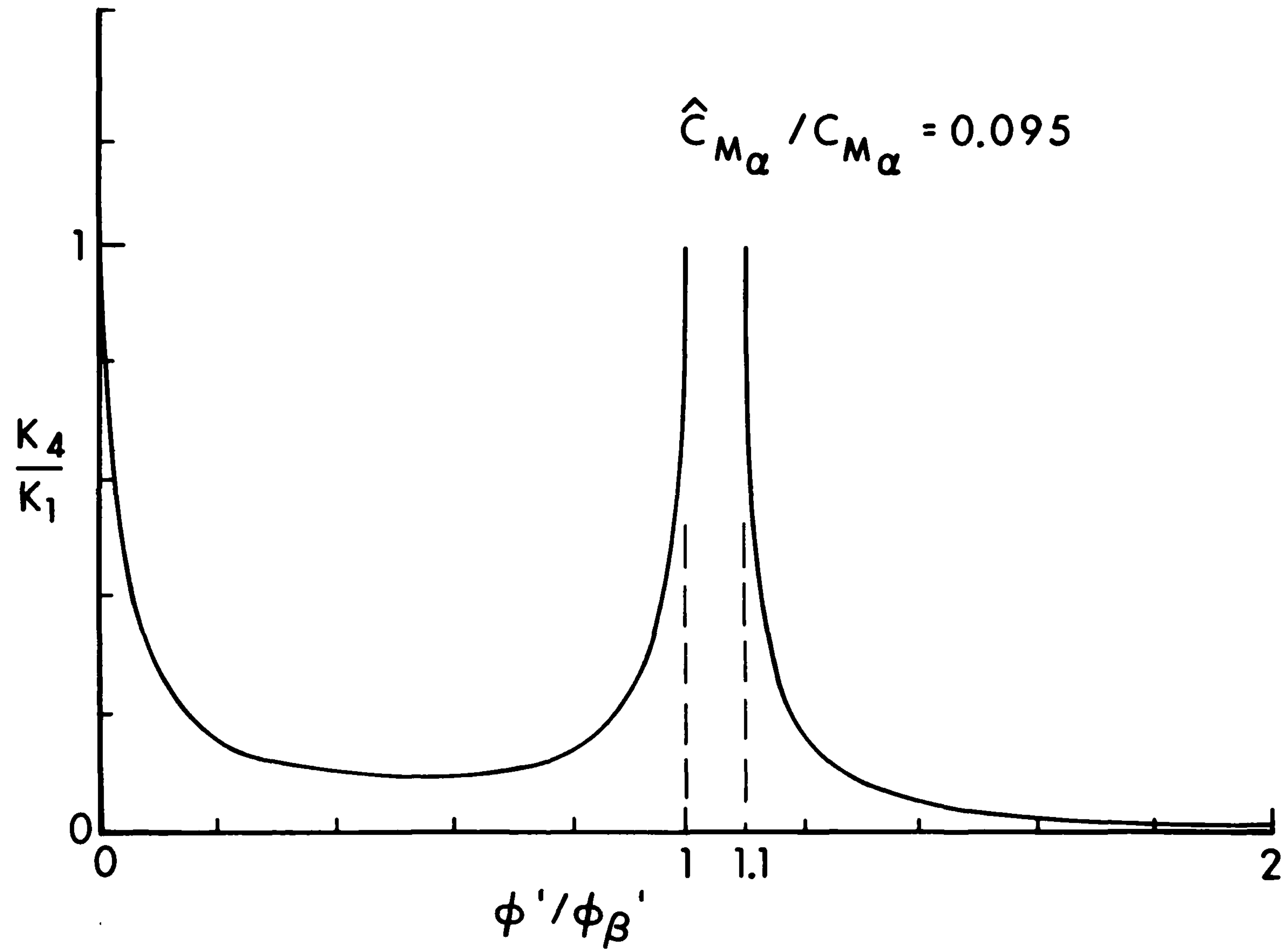

Figure 3. Amplitude Ratio K_4/K_1 versus Spin Rate ϕ'/ϕ'_β

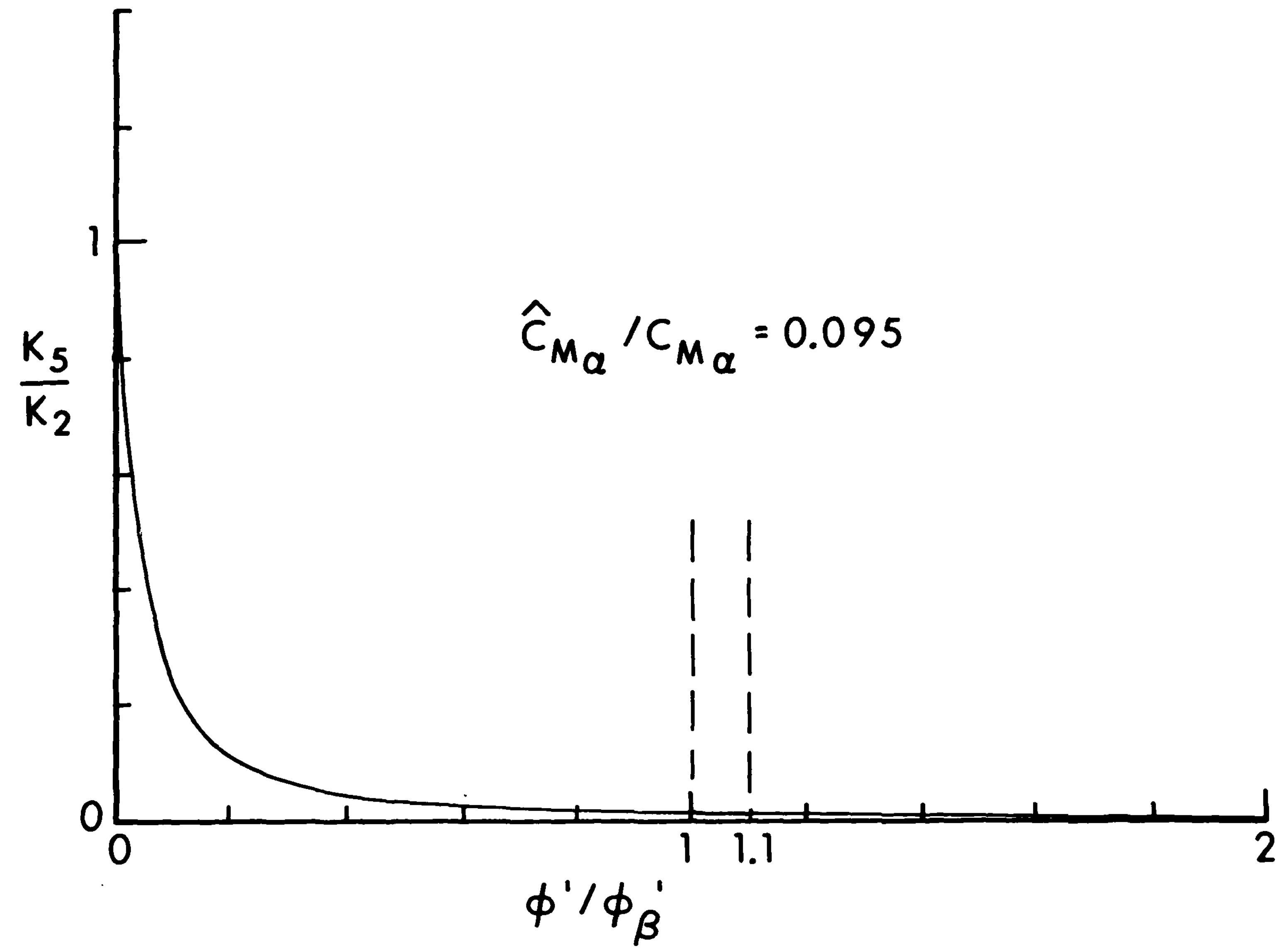

Figure 4. Amplitude Ratio K_5/K_2 versus Spin Rate ϕ'/ϕ'_β

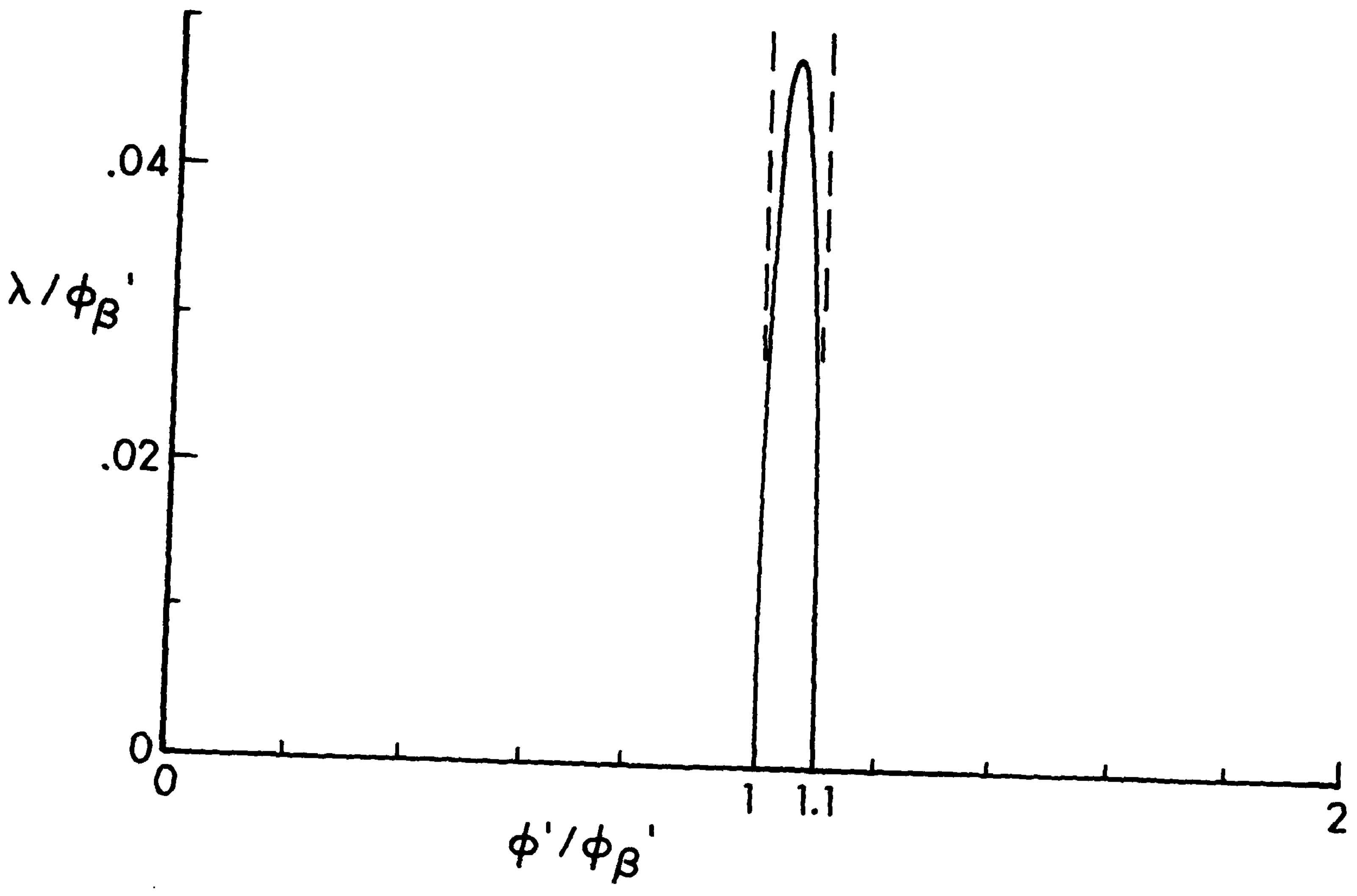

Figure 5. *Resonance Damping Rate* λ/ϕ_β' versus Spin Rate ϕ'/ϕ_β'

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LIST OF SYMBOLS

A	$k_t^{-2} (C^*_{m_0} + i\, C^*_{n_0})$
a	$(f\, \phi')^2/2 - M_r$
b	$M_r^2 - \hat{M}^2$
C_D	$\dfrac{\text{drag force}}{\rho\, S\, V^2/2}$
C_ℓ, C_m, C_n	missile-fixed components of $\left(\dfrac{\text{aerodynamic moment}}{\rho\, S\, \ell\, V^2/2}\right)$
C_X, C_Y, C_Z	missile-fixed components of $\left(\dfrac{\text{aerodynamic force}}{\rho\, S\, V^2/2}\right)$
C_1, C_2	$\omega_j^2 - f\, \phi'\, \omega_j + M_r\,$, $j = 1, 2$
$C_{(\ )}$, $\hat{C}_{(\ )}$	see Table IV for all C definitions not given above
f	$2 - (I_x/I_t)$
g_1	$-\omega_{2S}\, [2\, f\, \phi'\, M_r\, (\omega_{1S} - \omega_{2S})]^{-1}$
g_2	$\omega_{1S}\, [2\, f\, \phi'\, M_r\, (\omega_{1S} - \omega_{2S})]^{-1}$
H	$C^*_{N_\alpha} - 2\, C^*_D - k_t^{-2}\, (C^*_{M_q} + C^*_{M_{\dot\alpha}})$
$\hat{H}$	$\hat{C}^*_{N_\alpha} + k_t^{-2}\, (\hat{C}^*_{M_q} + \hat{C}^*_{M_{\dot\alpha}})$
$\vec{H}$	angular momentum
I_t	transverse moment of inertia ($= I_y = I_z$)
I_x	axial moment of inertia
K_j	absolute value of k_j
K_{jR}	absolute value of k_{jR}

k_j	complex yaw modes, $j = 1\text{-}5$
k_{jR}	complex yaw modes in the resonance region, $j = 1, 2$
k_t^2	$I_t/m\ell^2$
ℓ	reference length
M	$k_t^{-2}\ C^*_{M_\alpha}$
$\hat{M}$	$k_t^{-2}\ \hat{C}^*_{M_\alpha}$
M_r	$M + (f - 1)(\phi')^2$
M_S	$P(C^*_{N_\alpha} - C^*_D) + k_t^{-2}\ C^*_{MS_\alpha}$
$\hat{M}_S$	$f\ \phi'\ \hat{C}^*_{N_\alpha}$
m	mass
P	$(I_x/I_t)\phi'$
p	spin rate (rad/s)
q, r	transverse angular velocity components in the missile-fixed system
S	reference area
s	dimensionless arclength along the trajectory
t	time
u, v, w	missile-fixed components of $\vec{V}$
$\vec{V}$	missile velocity
V	magnitude of $\vec{V}$
X, Y, Z	missile-fixed axes in which the X-axis is along the principal axis of inertia nearest to the flight direction

α	$\tan^{-1}$ (w/u), the angle of attack
β	$\sin^{-1}$ (v/V), the angle of sideslip
δ	$\begin{cases} 1 \text{ if } \phi' > \phi'_\alpha \\ -1 \text{ if } 0 \le \phi' < \phi'_\beta \end{cases}$
ϵ	any positive number small in comparison with $\phi'_\alpha - \phi'_\beta$
λ	exponential damping coefficient in the resonance-region solution (5.1) of the yaw equation (4.1)
λ_1, λ_2	exponential damping coefficients in the solution (7.4) of the yaw equation (7.3)
λ_{jS}	$-(H\phi'_j - M_S)/(2\phi'_j - P)$, $j = 1, 2$ (the symmetric-missile value of λ_j)
μ	$(q + ir)\ell/V$
ξ	$\frac{v + i w}{V}$, missile-fixed complex angle of attack
$\tilde{\xi}$	$\xi e^{i\phi}$, aeroballistic complex angle of attack
ρ	air density
ϕ	$\phi' s$, the roll angle
ϕ_{jR}	argument of k_{jR}
ϕ_{j0}	argument of k_j
ϕ'	$p\ell/V$ = spin rate (rad/cal)
ϕ'_j	j-th yaw arm frequency in the aeroballistic system, $j = 1, 2$
ϕ'_{jS}	$\left(P \pm \sqrt{P^2 - 4M}\right)/2$, $j = 1, 2$ (the symmetric-missile value of ϕ'_j)
ϕ'_α	upper boundary of the resonance spin region

ϕ'_β	lower boundary of the resonance spin region
ω_j	j-th yaw arm turning rate in the missile-fixed system, $j = 1, 2$
ω_α	$(-k_t^{-2} \; C^*_{m_\alpha})^{1/2}$, the pitch frequency for zero spin
ω_β	$(k_t^{-2} \; C^*_{n_\beta})^{1/2}$, the yaw frequency for zero spin
$\vec{\omega}$	angular velocity of the missile-fixed system with respect to an inertial system

Superscripts

$(\dot{\ })$	$= d(\ )/dt$
$(\ )'$	$= d(\ )/ds = (\dot{\ })\ell/V$
$(\ )^*$	$= \dfrac{\rho S \ell}{2m}(\ )$
$(\bar{\ })$	= complex conjugate of ( )

APPENDIX A. EFFECT OF SMALL MASS ASYMMETRY

In this report, we have restricted the allowable asymmetries to the aerodynamic force and moment. Mass asymmetries can change the center of mass and the inertia properties. A transverse shift of the center of mass introduces a trim moment due to drag and a roll moment due to lift; these moments can be easily handled.[12] The effects of unequal transverse moments of inertia and of products of inertia are more complicated and will be considered here. We will outline the procedure to handle small inertia asymmetries and show their impact on the results of this report.

For asymmetric inertia properties, the angular momentum vector has missile-fixed components

$$H_x = I_x\, p - J_{xy}\, q - J_{xz}\, r \tag{A1}$$

$$H_y = - J_{xy}\, p + I_t[(1 + d_1)\, q + d_2 r] \tag{A2}$$

$$H_z = - J_{xz}\, p + I_t\, [d_2 q + (1 - d_1)\, r] \tag{A3}$$

where

$$d_1 = (I_y - I_z)/(2\, I_t)$$

$$d_2 = - J_{yz}/I_t$$

$$I_t = (I_y + I_z)/2$$

12. *D.A. Price, Jr., "Sources, Mechanisms, and Control of Roll Resonance Phenomena for Sounding Rockets," Journal of Spacecraft and Rockets 4, November 1967, pp. 1516-1525.*

and

$$J_{xy}, J_{yz}, J_{xz} \quad \text{are products of inertia.}$$

Equation (2.13) now takes on the form

$$\mu' + i\,(\phi' - P)\mu + (d_1 + i\,d_2)(\bar{\mu}' + i\,\phi'\,\bar{\mu})$$

$$= k_t^{-2}\,(C_m^* + i\,C_n^*) \tag{A4}$$

$$+ C_D^*\,[\mu + (d_1 + i\,d_2)\bar{\mu}]$$

$$+ i\,(J_{xy} + i\,J_{xz})(\phi')^2/I_t$$

C_D can now be neglected and the static moment of Equation (3.5) can be inserted to obtain a new form of Equation (4.1):

$$\xi'' + i\,f\,\phi'\,\xi' - M_r\,\xi + \hat{M}\,\bar{\xi} \tag{A5}$$

$$- (d_1 + i\,d_2)[\bar{\xi}'' + (\phi')^2\bar{\xi}] = i\,A$$

where

$$A = k_t^{-2}\,(C^*_{m_0} + i\,C^*_{n_0}) + i\,(J_{xy} + i\,J_{xz})(\phi')^2/I_t$$

If the pentacycle solution, Equation (4.4), is substituted in Equation (A5), we obtain the following relations for the modal amplitudes:

$$k_3 = -\,i\left[M_r\,A - [\hat{M} - (d_1 + i\,d_2)(\phi')^2]\bar{A}\right]b^{-1} \tag{A6}$$

$$k_4 = \left[\hat{M} + (d_1 + i\ d_2)[\omega_1^2 - (\phi')^2]\right] \bar{k}_1\ C_1^{-1} \tag{A7}$$

$$k_5 = \left[\hat{M} + (d_1 + i\ d_2)[\omega_2^2 - (\phi')^2]\right] \bar{k}_2\ C_2^{-1} \tag{A8}$$

$$C_j^2 + 2\ f\ \phi'\ \omega_j\ C_j - \left[\hat{M} + d_1\ [\omega_j^2 - (\phi')^2]\right]^2 - d_2^2\ [\omega_j^2 - (\phi')^2]^2 = 0 \tag{A9}$$

where

$$C_j = \omega_j^2 - f\ \phi'\ \omega_j + M_r$$

$$b = M_r^2 - [\hat{M} - d_1\ (\phi')^2]^2 - d_2^2\ (\phi')^4$$

It should be noted that for general motion away from zero spin or the resonance region, a good approximation for C_j is still $-\ 2\ f\ \phi'\ \omega_j$.

The new frequency equation can be obtained from Equation (A9) by eliminating C_j and assuming small mass asymmetry (i.e. neglecting d_j^2 in comparison with one).

$$\omega_j^4 - 2\ a\ \omega_j^2 + b = 0 \tag{A10}$$

where

$$a = (f\phi')^2/2 - M_r + \hat{M}\ d_1 - (d_1^2 + d_2^2)(\phi')^2$$

The pitch and yaw frequencies for zero spin are given by

$$\omega_\alpha^2 = -\ k_y^{-2}\ C^*_{m_\alpha} \tag{A11}$$

$$\omega_\beta^2 = k_z^{-2}\ C^*_{n_\beta} \tag{A12}$$

where

$$k_y = \sqrt{I_y/m\ \ell^2}$$

$$k_z = \sqrt{I_z/m\ \ell^2}$$

The coefficients of the frequency equation can be expressed in terms of these zero-spin frequencies.

$$a = \frac{\omega_\alpha^2 + \omega_\beta^2 + (\phi')^2(f^2 - 2\ f + 2)}{2} \tag{A13}$$

$$b = \left[\omega_\alpha^2 - \left(\frac{I_z - I_x}{I_y}\right)\ (\phi')^2\right]\left[\omega_\beta^2 - \left(\frac{I_y - I_x}{I_z}\right)(\phi')^2\right]$$

$$-\ d_2^2\ (\phi')^4 \tag{A14}$$

The resonance spin region is bounded by the zeros of b. For no product of inertia ($d_2 = 0$), we see that these zeros are

$$\phi'_\alpha = \left[I_y/(I_z - I_x)\right]^{1/2} \omega_\alpha \tag{A15}$$

$$\phi'_\beta = \left[I_z/(I_y - I_x)\right]^{1/2} \omega_\beta \tag{A16}$$

In summary, we see that asymmetric mass terms complicate the algebra but do not change the four conclusions of the report.

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